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ABSTRACT

This is one of a series of 20 booklets designed for participants in an in-service course for teachers of elementary mathematics. The course, developed by the University of Illinois Arithmetic Project, is designed to be conducted by local school personnel. In addition to these booklets, a course package includes films showing mathematics being taught to classes of children, extensive discussion notes, and detailed guides for correcting written lessons. This booklet contains exercises on the effects of using rules in different orders, a summary of the problems in the film "Three A's, Three B's, and One C," and the supplement. (MK)

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THE ARITHMETIC PROJECT COURSE FOR TEACHERS

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TOPIC: Effects of Using Rules in Different Orders.

FILM: Three A's, Three B's, and One C, Grade 5

SUPPLEMENT: Dividing by Zero

4

NAME: .....

SE 031 104

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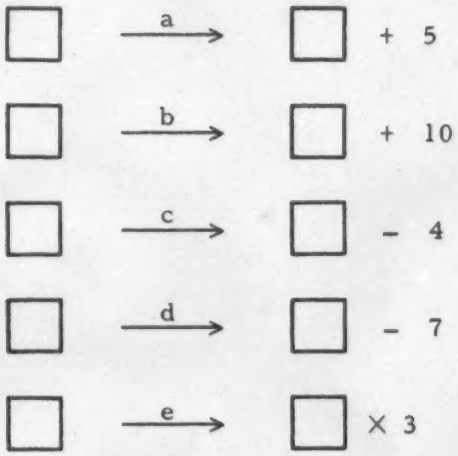
BOOK FOUR

I.

These are the jumping rules that apply to questions on pages 1 through 5.



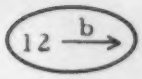
Some of the following problems have been answered to guide you. Do the others:

1. 6 —a→ = 11

2. 11 —b→ = 21

3. 6 —a→ —b→ = 21

4. 7 —a→ = _____

5. 

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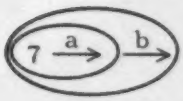
6. 

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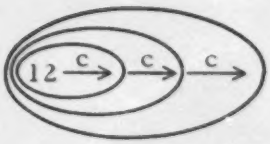
7. 

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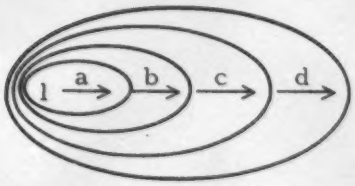
8. 

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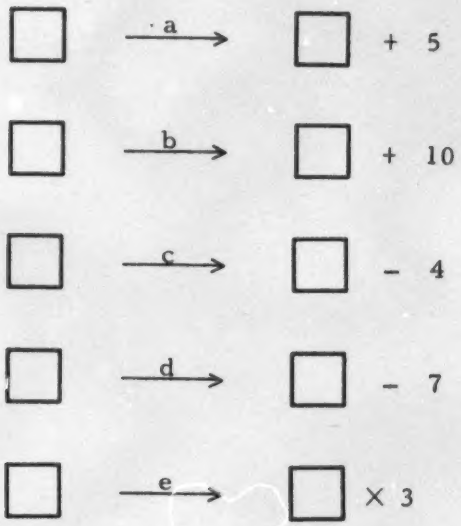
9. Sometimes the incorrect answer "4" is given to the problem above. This might be due to an error in computation, but more often than not the error is probably the result of something else. What?

This page may be removed for convenient reference to the rules given on page 1; these rules apply to problems on pages 1 through 5.



tear here

We can also use the following shortened code:

$20 \xrightarrow{aa}$ will mean the same thing as $20 \xrightarrow{a} \xrightarrow{a}$

10. $20 \xrightarrow{aa} =$ _____

Arithmetic problems:

11. $20 \xrightarrow{aa} - 20 \xrightarrow{b} =$ _____

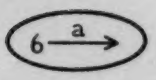
12. $5 \xrightarrow{ec} + 13 \xrightarrow{da} =$ _____

13. $5 \xrightarrow{c} \times 2 \xrightarrow{b} \times 8 \xrightarrow{d} \times \frac{1}{3} \xrightarrow{e} =$ _____

14. The most likely wrong answer to #13 is 20. Comment?

15. $1{,}467 \xrightarrow{dccabe} \times 7 \xrightarrow{d} =$ _____

II.

1. 

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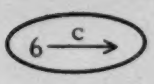
2. 

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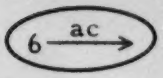
3. 

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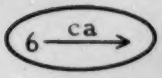
4. 

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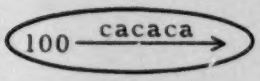
5. 

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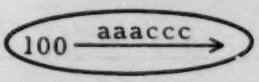
6. 

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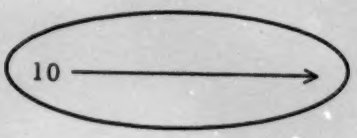
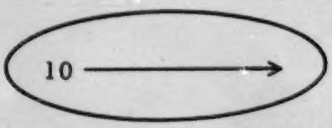
7. Given three a's, three c's and one e, put them down so as to get the largest possible number:



What is the number? _____

8. What is the next question? _____

9. Give the answer: 

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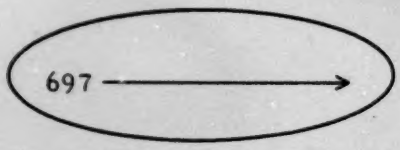
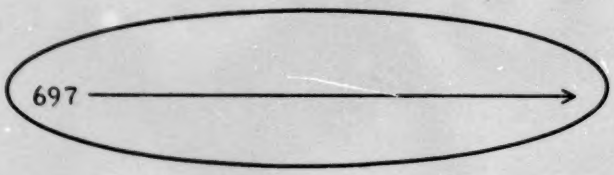
10. Given two a's, two b's, two c's, and two d's, how should you put them down to get the largest possible number?



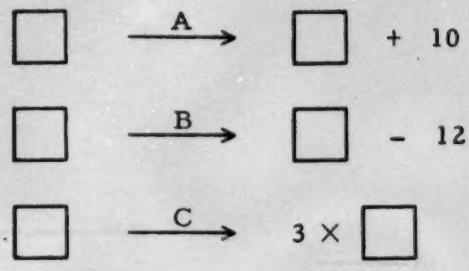
What is the number? _____

Comment:

11. Given ten b's, ten c's, and ten e's, how should you put them down to get the largest possible number?



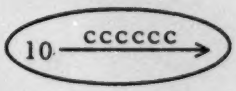
(Never mind finding the number.)

12. New rules: 

When three A's, three B's, and one C are used on a certain number, the highest number that can be obtained is 207. What is the lowest number that can be obtained? _____

III.

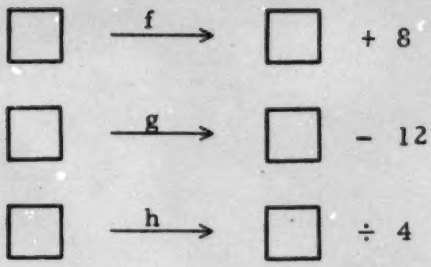
In the film, at a place where rule c is □ $\xrightarrow{c}$ 2 × □ , the problem



is given. The children's answers were predominantly 22 and 120. What are likely routes by which the children arrived at each of these answers?

10

IV.



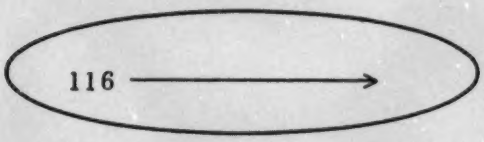
1. 24 —fgh→ =

2. 24 —fhg→ =

3. 24 —ghf→ =

4. 24 —hfg→ =

5. Given three f's, three g's, and one h, put them down so as to get the largest possible number in:

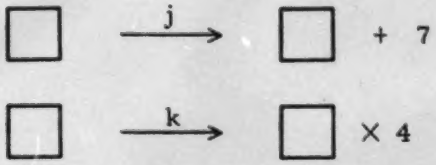


What is the number? _____

V.

So far we have not considered any questions dealing with how big the difference may be between the largest possible number and the smallest when rules are used in different orders.* Could one, for example, predict how big the difference will be just by looking at the rules?

To investigate, consider two rules:



1. Start at 13. You know that $13 \xrightarrow{jk}$ is larger than $13 \xrightarrow{kj}$.

   By how much? _____

2. Start at 0.

   $0 \xrightarrow{jk}$ = _____

   $0 \xrightarrow{kj}$ = _____

Larger number minus smaller number equals _____

* This question is related to the question at the end of the film, "Can you get all the (whole) numbers in between?"

Continue with the rules: □ $\xrightarrow{j}$ □ + 7 and □ $\xrightarrow{k}$ □×4

3. $-10 \xrightarrow{jk}$ = _____

$-10 \xrightarrow{kj}$ = _____

Larger minus smaller equals 21

(The larger number is _____)

4. Now change the adding rule, but keep the multiplying rule the same:

□ $\xrightarrow{m}$ □ + 8

□ $\xrightarrow{k}$ □ × 4

Maximum using each rule once and starting at 13:
(Write letters inside loops.)

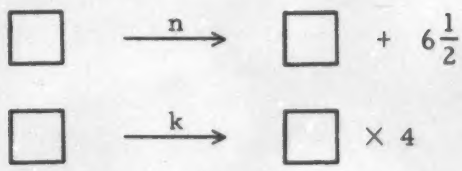
$13 \longrightarrow$ = _____

Minimum:

$13 \longrightarrow$ = _____

Maximum minus minimum: _____

5. Change adding rule again:

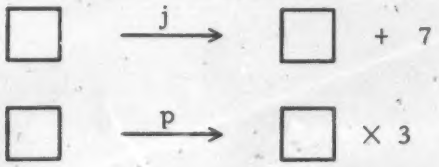


How much will the winning combination, kn or nk, win by (maximum minus minimum) if you start at $1\frac{1}{2}$? _____

At 133 ? _____

(Be sure that you wrote the difference, and not the winning combination.)

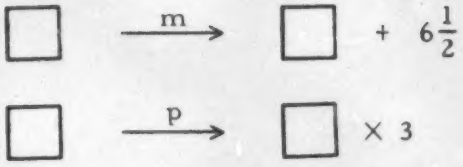
6. Now go back to the original rule j, but change the multiplying rule:



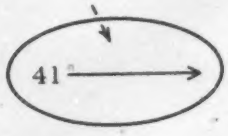
13 —pj→ = _____

13 —jp→ = _____ Difference: _____

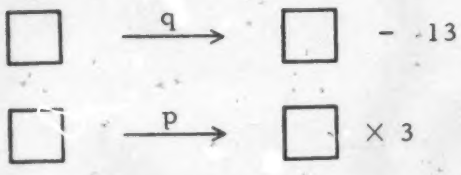
What is happening?

7. 

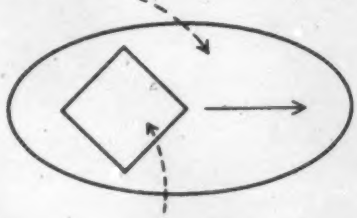
Write letters for maximum



wins by _____

8. 

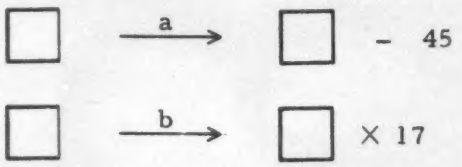
Write letters



wins by _____

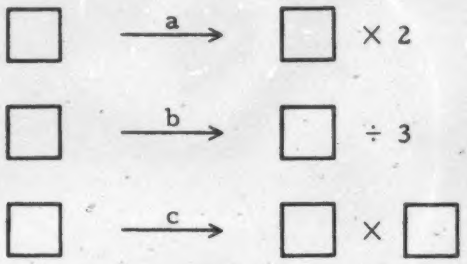
What number do you pick for the diamond? _____

☆9. Tell how you would predict by how much the "better" combination will win when you start at 293:

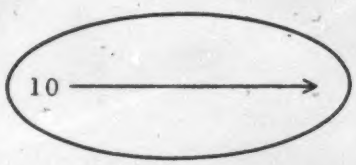


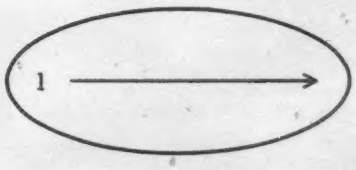
☆VI.

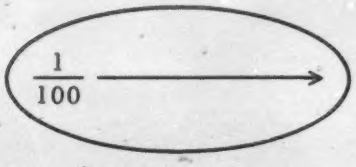
Instead of three rules which add, subtract and multiply, consider three rules which multiply, divide, and square:



Using one a, one b, and one c, determine the order of the three letters to find the maximum in the following:

☆1. 

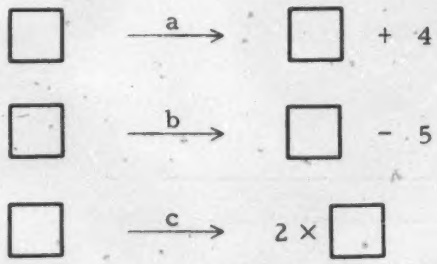
☆2. 

☆3. 

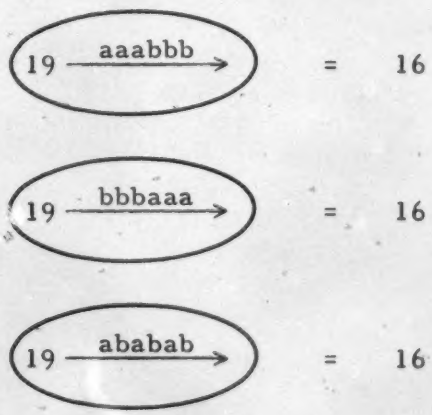
Epilogue

This lesson and the film you saw in connection with it deal with several mathematical ideas. One of these is the property called commutativity. One may know that addition is commutative (i.e., $13 + 95 = 95 + 13$) and that multiplication is commutative (e.g., $47 \times 191 = 191 \times 47$) but still not appreciate commutativity in other contexts. School children upon learning that $3 + 2$ gives the same answer as $2 + 3$ and that this exemplifies that addition is commutative often think, "So what else is new?"

Recall that in the film the rules were:



The students learned correctly that if the jumping rules used were only a and b, then the order of using them did not change the answer. For example,



and so on.

Then when rule c was introduced they expected that all three rules would commute. When they saw that $53 \xrightarrow{abababcb}$ gave a result different from $53 \xrightarrow{abababc}$ the students learned that rule c does not commute with a or b .

One fails to appreciate the idea of commutativity unless at some time he expects it to hold and it does not. Thus the children in the film expected that $53 \xrightarrow{abababc}$ would give the same result as $53 \xrightarrow{ababacb}$. When they found out that the results were different, they had gained some insight into the notion of commutativity.

Another important idea in mathematics is that of maximum and minimum. As soon as the class saw that the result for $53 \xrightarrow{abababc}$ changed when the letters were put down in a different order, a natural next question came up: If we put the seven letters down in all possible orders, which order gives the largest number as a result? Trying all possible orders would take far too long. The students were able to determine the order for a maximum by thinking about how adding 4, subtracting 5 and doubling are related.

You will see more maximum and minimum problems as this course continues.

Summary of Problems in the Film "Three A's, Three B's, and One C"

5th Grade, James Russell Lowell School, Watertown, Massachusetts
Teacher: David A. Page

We'll use several rules.

Rule a: $\square \xrightarrow{a} \square + 4$

Using rule a, start at 2 and make one jump. (Land: 6)

Start at 9 with rule a and make one jump.

Start at $1\frac{1}{2}$ and make one jump with rule a. Where do you land? ($5\frac{1}{2}$)

Start at 10 and make two jumps, one right after the other with rule a. (Land: 18)

Start at 19 and make two jumps.

Start at 23 and make 100 jumps with rule a. Where do you land? (423)

Rule b: $\square \longrightarrow \square - 5$

Start at 13, use rule b and make one jump. Land?

Start at 8 with rule b. (Land: 3)

Start at 3, use rule b and make a jump.

Another jump with rule b from -2. Land?

Another jump from -7; now one from -12.

Another jump with rule b starting at -17. (Land: -22)

Still using rule b, make two jumps, one right after the other, from -22.

Now start at -32 and make 20 jumps, one right after the other.

Start at 7 and make a jump with rule a and then a jump with rule b.

Where do you land? (6)

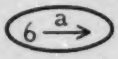
Rule c: $\square \xrightarrow{c} 2 \times \square$

Start at 5 with rule c. Land?

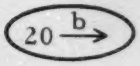
Another jump from 10 with rule c. Where do you land?

Start at $4\frac{1}{2}$ and make one jump with rule c. (Land: 9)

Using these rules, I'm going to give you a code. This stands for a number:



What is the number? (10)



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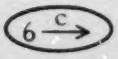

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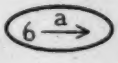
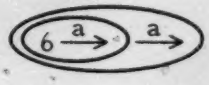
You know that 

= 10 , so now what is this:



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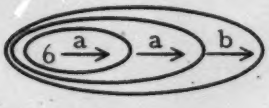
And since you know what this is, now what is this:



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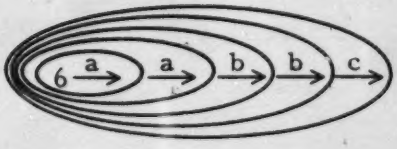

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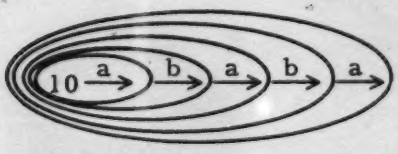

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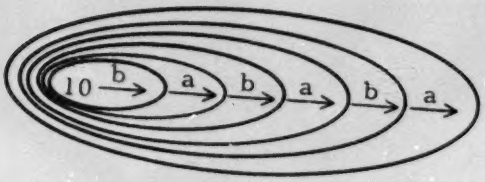

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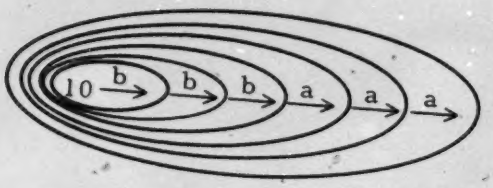
What is a fast way to do the problem above?



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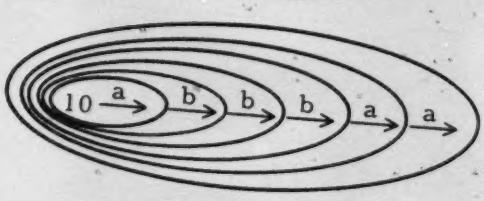

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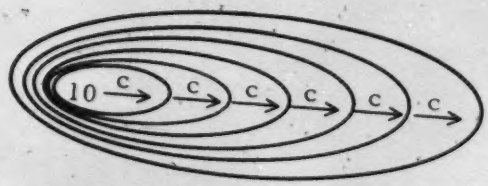

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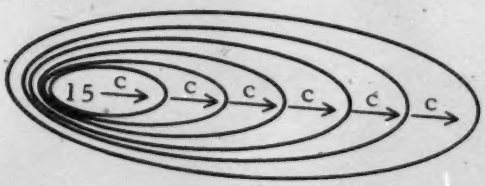

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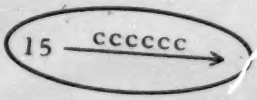
We'll write this a shorter way:



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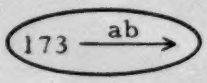
More problems:



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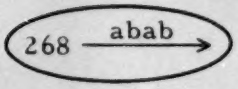

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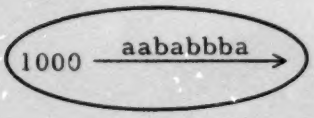

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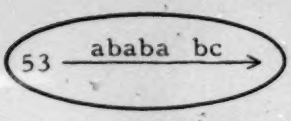

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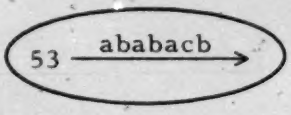
New problem: Pick up the c and put it over here in this slot.
Now what is the answer:



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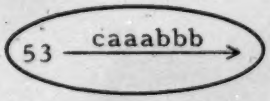
Putting the c in different places seems to change things. So here is your problem: Start at 53; you can have three a's and three b's and one c. How should you put them down to get the biggest number that you can?



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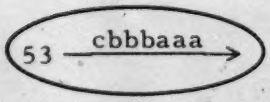

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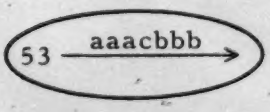

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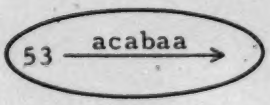

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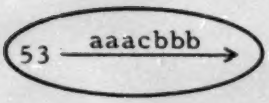
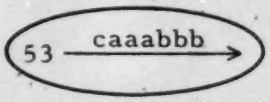
(No fair: What is wrong with this problem?)

Why is 

likely to be the biggest?

New question: How shall we put three a's, three b's, and one c down in order to get the smallest possible number?



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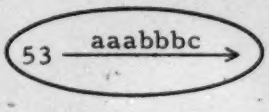
We did this one a long time ago: 

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The biggest is 115 and the smallest is 88. Here is a homework problem: Can you get all the whole numbers in between 115 and 88, or just some, and which ones?

Supplement

Dividing by Zero

I.

One of the common hazards in teaching arithmetic is the question of dividing by zero. (The consternation of some conscientious teachers may even be pursued with delight by unscrupulous children.) There is no need for fearfulness, however, as the question can be, and should be, dealt with firmly.

Look at the following:

$$8 \times \frac{1}{8} = 1$$

$$17 \times \frac{1}{17} = 1$$

$$\frac{2}{5} \times \frac{5}{2} = 1$$

8 and $\frac{1}{8}$, 17 and $\frac{1}{17}$, $\frac{2}{5}$ and $\frac{5}{2}$ are all related in a special way——they multiply to give 1. By definition two numbers are <u>multiplicative inverses</u> of each other if their product is 1. Thus one-eighth is the multiplicative inverse of eight, and eight is the multiplicative inverse of one-eighth.

<u>Question:</u> What is the multiplicative inverse of 0?

By the definition of a multiplicative inverse we know that 0 times its multiplicative inverse must equal 1. But we also know that 0 times any number is 0. So we seem to have a contradiction. Either $0 \times \square = 0$ must be false for some number in the box, or else 0 does not have a multiplicative inverse. In order to retain the properties and common rules of arithmetic we are forced to conclude that 0 does not have a multiplicative inverse.*

---

* The properties that are at stake here are:

1) For all numbers b, $0 + b = b$
2) For each number b, there is a number $-b$ such that $-b + b = 0$
3) For all numbers b, c and d, $(b + c) + d = b + (c + d)$
4) For all numbers b, c and d, $(b + c) \times d = b \times d + c \times d$

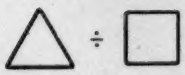
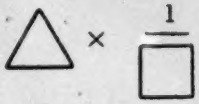
Here are some problems using inverses:

$30 \div 6 = 30 \times \frac{1}{6} =$

$22 \div 9 = 22 \times$ $= \frac{22}{9}$

$7 \div \frac{1}{5} = 7 \times$ $= 35$

$0 \div 3 =$ $\times \frac{1}{3} = 0$

 $=$  $=$

In general it is true that dividing △ by □ is the same as multiplying △ by the multiplicative inverse of □. (This is a definition of division.) However, we cannot put 0 in the box, since, as we have seen, 0 does not have a multiplicative inverse. Therefore, $\triangle \div \square = \triangle \times \frac{1}{\square}$ is always true except when you put 0 in the box, because $\frac{1}{0}$ does not make sense. Division by 0 is against the rules.

II.

We can look at attempting to divide by zero a little differently. Consider the problem $8 \div 4 = \square$. Since division and multiplication are inverse operations, $8 \div 4 = \square$ is equivalent to $\square \times 4 = 8$. Similarly,

$30 \div 6 = 5$	because	$5 \times 6 = 30$
$7 \div \frac{1}{5} = 35$	because	$35 \times \frac{1}{5} = 7$
$0 \div 3 = 0$	because	$0 \times 3 = 0$

If we write $10 \div 0 = \square$, proceeding as above, we would like to know what works in the box for $\square \times 0 = 10$. Since $\square \times 0 = 0$, no matter what number is used in □, there can be no number that makes $\square \times 0 = 10$ true.

Therefore, there can be no number which makes $10 \div 0 = \square$ a true statement. Division by 0 cannot be done.

III.

On occasion a teacher is confronted by a student whose father (an engineer, or scientist, perhaps) has said that 32 divided by zero is infinity. (Write "∞" on the blackboard at your own peril!) This father was summarizing a profound idea that comes up in scientific and engineering work. Consider the following sequence of equations:

$$32 \div 8 = 4$$

$$32 \div 2 = 16$$

$$32 \div 1 = 32$$

$$32 \div \frac{1}{2} = 64$$

$$32 \div \frac{1}{30} = 960$$

$$32 \div \frac{1}{100} = 3{,}200$$

$$32 \div \frac{1}{1{,}000{,}000} = 32{,}000{,}000$$

If we let d stand for the number we are dividing by, we can say that d in the equations above is approaching 0. The closer the number d is to 0, the larger the answer will be. In scientific work it is often said that as a positive number d approaches 0, $32 \div d$ approaches infinity. This statement is usually abbreviated: $\lim_{d \to 0^+} \frac{32}{d} = \infty$

There is nothing special about 32. Whatever positive number you are given, if you divide it by a sufficiently small positive number you can make the result as large as you like. As the divisor gets closer and closer to 0, the result grows beyond all bounds. No number on the number line has the quality of unboundedness. (If one of your students majors in mathematics in college and then goes on to graduate work in mathematics, he will see several kinds of infinities dealt with in sensible——if complicated——ways.)

IV.

A physical model illustrating the impossibility of dividing by 0 is provided by the following problem: Suppose that I have 240 pencils at the back of the room and pencil boxes to carry them to the front of the room. If I put 24 pencils in each box, how many boxes will I need to get all the pencils to the front of the room? _____

If I put 12 in each box, how many boxes do I need? _____

10 in each box; how many boxes? _____

6 in each box; how many boxes? _____

3 in each box; how many boxes? _____

2 in each box; how many boxes? _____

1 in each box; how many boxes? _____

$\frac{1}{3}$ in each box; how many boxes? _____

0 in each box; how many boxes? _____

Proper answers to the last question are "Nonsense!" or "Don't be silly!" "Nonsense!" and "Don't be silly!" are what should be said to $\frac{240}{0}$.

Having heard that dividing by zero is not permissible, some teachers become afraid to use zero at all. All of the following are perfectly satisfactory numbers in arithmetic.

$0 \div 2$ $\qquad$ $\frac{0}{16\frac{1}{2}}$ $\qquad$ 10×0

Each of these is equal to 0.

For those who would like some practice computing with 0, here are a few problems. Find all the numbers that work. Answers are on the next page.

1. $1{,}000{,}000 \times 0 = \triangle$

2. $0 \div \frac{1}{10} = $ _____

3. $14 + \square = 14$

4. $3 + 3 + \triangle + 3 = 0$

5. $0 \times (\square + \triangle) = 0$

(Remember that if you have frames of different shapes in an equation you may use different numbers or the same number in different shapes. For example, in $\square + \square + \triangle = 9$ you may have

$2 + 2 + 5 = 9$

or $3 + 3 + 3 = 9$

or $4\frac{1}{2} + 4\frac{1}{2} + 0 = 9$, etc.)

6. $41 \times \square + 3 = 3$

7. $\triangle \div \frac{1}{2} = 2 \div \triangle$

8. $\square \div (\square + \triangle) = \triangle - \triangle$

9. $\dfrac{\square - \triangle}{\square + \triangle} = (\square - \square) \times (\square + \triangle)$

Answers to the problems on page 25.

1. 0

2. 0

3. 0

4. -9

5. Any numbers for $\square$ and $\triangle$.

6. 0

7. 1 or -1 (If you thought 0 would work, look at the right side of the equation.)

8. $\square = 0$, $\triangle$ = any number except 0.

9. $\square = \triangle$ but neither $\square$ nor $\triangle$ can be 0.

One way to solve problem 9 is to look at the right side and see $(\square - \square)$. No matter what number is in the box, $\square - \square$ will be 0, and 0 times $(\square + \triangle)$ will be 0. So the right side of the equation is 0. We have

$$\frac{\triangle - \square}{\square + \triangle} = 0$$

If the equation in problem 9 is to be true, then the left side must be 0 and $\triangle - \square$ must be 0. For $\triangle - \square$ to be 0, $\triangle$ must equal $\square$. However, $\triangle$ and $\square$ cannot both be 0 because this would mean that $\square + \triangle$ would be 0. The problem would then involve division by 0.